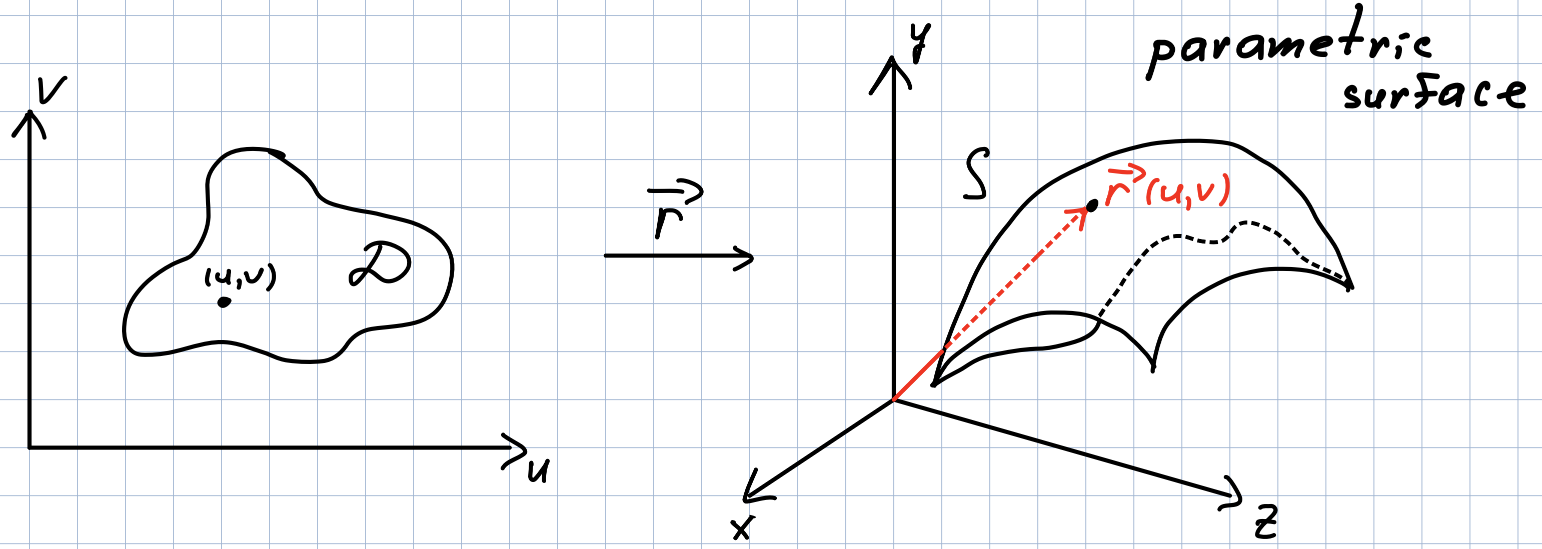
Last time:

Surface integrals:



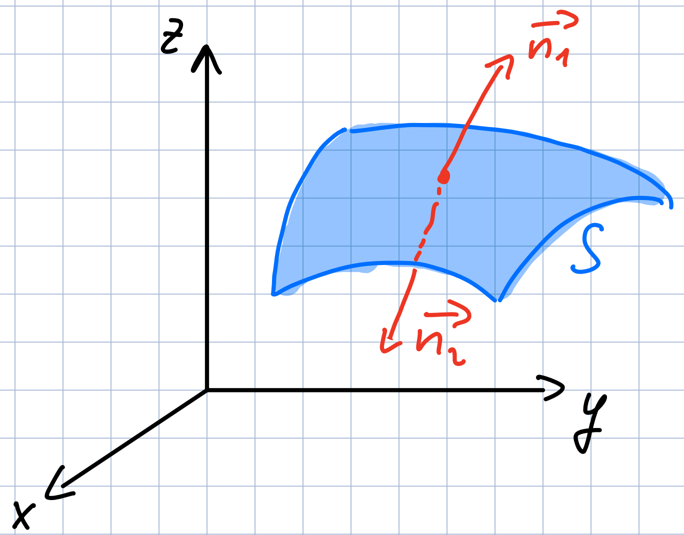
Surface integral of a function:

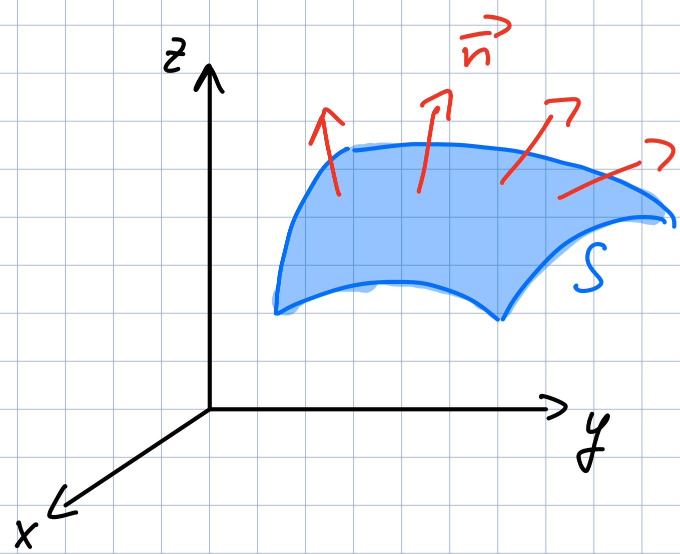
$$\iint_S f(x,y,z)\,dS = \iint_{\mathcal{D}} f(\vec{r}(u,v))\,\underbrace{|\vec{r}_u \times \vec{r}_v|\,\overbrace{dA}^{du\,dv}}_{dS}$$

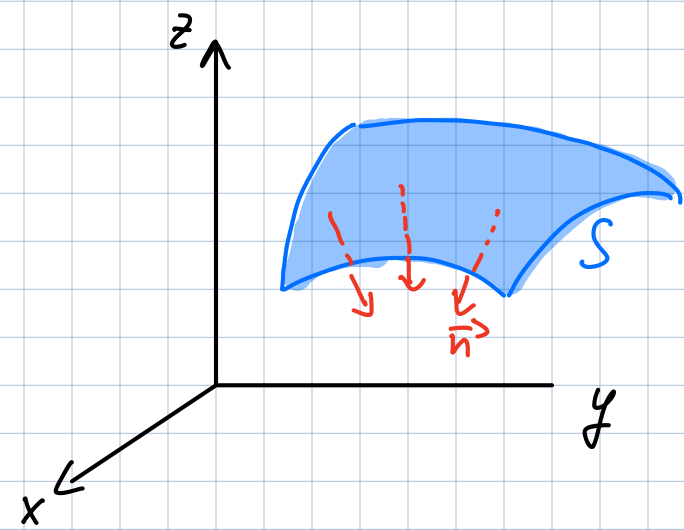
Oriented surfaces:

S - orientable (two-sided) surface.

- has a tangent plane at every point on S (except on boundary)







Flux integral

Flux integral is a surface integral of a vector field:

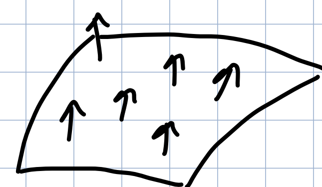
$$\iint_S \vec F \cdot d\vec S \overset{def}{=\!=} \iint_S \vec F \cdot \vec n \, dS = \iint_D \vec F(\vec r(u,v)) \cdot (\vec r_u \times \vec r_v) \underbrace{dA}_{du\,dv}$$

vector field $\vec F(x,y,z)$

unit normal vertor field for S

$$\vec n = \frac{\vec r_u \times \vec r_v}{|\vec r_u \times \vec r_v|}$$ - defines an orientation

(defoult orientation induced by parametrization)



Rmk: For $\vec n = -\dfrac{\vec r_u \times \vec r_v}{|\vec r_u \times \vec r_v|}$

$$\iint_S \vec F \cdot d\vec S = -\iint_D \vec F(\vec r(u,v)) \cdot (\vec r_u \times \vec r_v)\, dA.$$

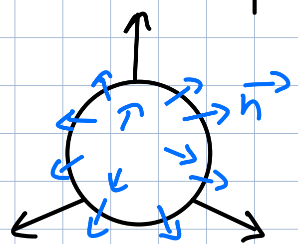
<u>Ex:</u> Find the flux of the vector field $\vec{F}(x,y,z) = \langle z, y, x \rangle$ across the unit sphere $x^2+y^2+z^2=1$.

<u>Sol:</u> 1) parametrize S by $\vec{r}(\varphi,\theta) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$

$$0 \le \varphi \le \pi$$
$$0 \le \theta \le 2\pi$$

2) $$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix} = \langle \sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi \rangle$$

$$= \sin\varphi \underbrace{\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle}_{\vec{n}\text{ - unit normal vector pointing outward}}$$



– "positive orientation"

3) $\vec{F}(\vec{r}(\varphi,\theta)) = \langle \cos\varphi, \sin\varphi\sin\theta, \sin\varphi\cos\theta \rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{\pi}\int_0^{2\pi} \left(\sin^2\varphi\cos\varphi\cos\theta + \sin^3\varphi\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}} + \sin^2\varphi\cos\varphi\cos\theta\right) d\theta\, d\varphi$$

$$= \int_0^{\pi} \pi\sin^3\varphi\, d\varphi = \pi\int_0^{\pi}(1-\cos^2\varphi)\sin\varphi\, d\varphi \underset{u=-\cos\varphi}{=} \pi\int_{-1}^{1}(1-u^2)\,du = \pi\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{1}$$

$$= \frac{4\pi}{3}$$

- Flux of $\vec{F} = \langle P, Q, R \rangle$ through the surface S of form $z = g(x,y)$ (graph of g)

$$\vec{r}(x,y) = \langle x, y, g(x,y) \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = \langle -g_x, -g_y, 1 \rangle$$

Therefore $$\iint_S \vec{F} \cdot \vec{dS} = \iint_{\mathcal{D}} \langle P, Q, R \rangle \cdot \langle -g_x, -g_y, 1 \rangle \, dA$$

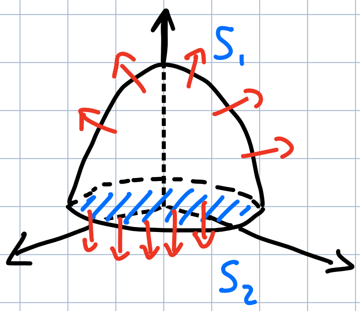
$\mathcal{D}$ in xy-plane

$$= \iint_{\mathcal{D}} (-Pg_x - Qg_y + R) \, dA$$

assuming upword orientation of S

Ex: S - boundary of the solid enclosed by $z = 1 - x^2 - y^2$ and $z = 0$.

$\vec{F} = \langle y, x, z \rangle$. Find the flux.



Choose outward (possitive) orientation.

Sol: 1) Flux through S_1:

$$\iint_{S_1} \vec{F} \cdot \vec{dS} = \iint_{x^2+y^2\le 1} \langle y, x, \underbrace{z}_{1-x^2-y^2} \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA = \iint_{x^2+y^2 \le 1} (\overbrace{2xy + 2xy}^{4xy} + 1 - x^2 - y^2) \, dA$$

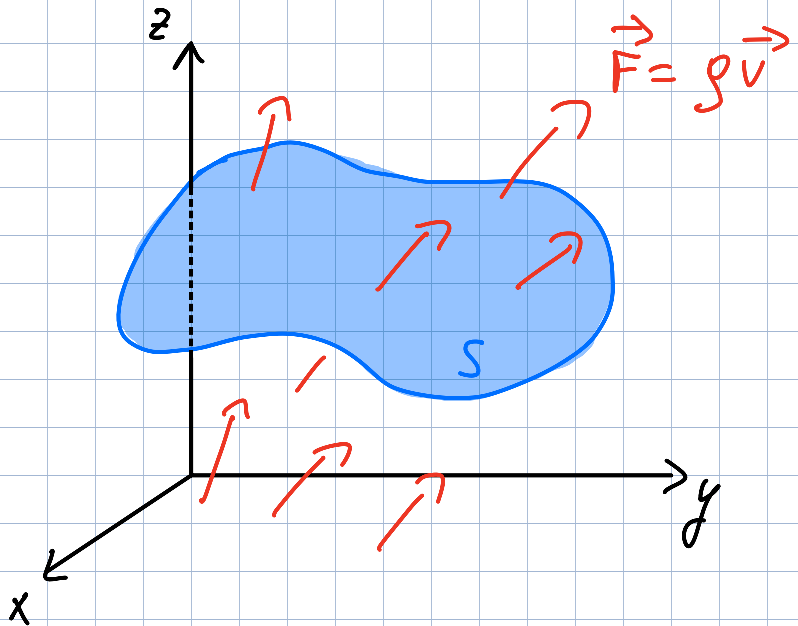
$$\underset{\text{polar}}{=\!=} \int_0^{2\pi} \int_0^1 (1 - r^2 + \underbrace{4r^2 \sin\theta\cos\theta}_{r^2 2\sin 2\theta}) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} r^2 - \frac{1}{4} r^4 - \frac{1}{2} \sin 2\theta \, r^4 \Big|_{r=0}^{r=1} \right) d\theta = \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4} - \frac{1}{2} \sin 2\theta \right) d\theta = \frac{\pi}{2}$$

2) $$\iint_{S_2} \vec{F} \cdot \vec{dS} = \iint_{x^2+y^2 \le 1} \langle y, x, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dA = 0$$

Thus total flux: $\displaystyle\iint_S \vec{F} \cdot \vec{dS} = \frac{\pi}{2}$

Physical interpretation:



S - oriented surface with unit normal vector $\vec{n}$

Imagin a fluid with density $\rho(x,y,z)$ and velocity field $\vec{v}(x,y,z)$ flowing through S.

Then the rate of flow (mass per unit time) per unit area is given by $\rho\vec{v}$.

// $\rho\vec{v}\cdot\vec{n}\,A(S_{ij})$ - mass of fluid per unit time crossing S_{ij} in the direction of $\vec{n}$ //

$\iint\limits_S \rho\vec{v}\cdot\vec{n}\,dS$ - rate of flow through S.